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ON THE MAXIMA AND MINIMA OF ALGEBRAIC POLYNOMIALS.

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1. Let $u = A_1x^n + A_2x^{n-1} + A_3x^{n-2} + \ldots + A_{n-1} = \phi x \ldots\ldots (1)$ The expression ϕx is read "function of x".

Suppose we wish to find a value of x such that if we substitute it in the polynomial (1), this polynomial, or u, will be a maximum, or the greatest possible; or a minimum, or the least possible, according to the conditions of the problem. To discover such values of x,—for there may be several maxima or minima of different orders,—we shall proceed as follows:

Suppose x to be increased or decreased by a small quantity h, so that we have, if we accent the u for this case,

$$u' = \phi(x+h), \text{ or } u' = \phi(x-h). \ldots\ldots\ldots (2)$$

Let each of these be developed according to the ascending powers of h, and let the result be denoted as follows:

$$u' = \phi x + \phi' xh + \phi'' x \cdot \frac{h^2}{1.2} + \phi''' x \cdot \frac{h^3}{1.2.3} + \ldots (3)$$

$$u' = \phi x - \phi' xh + \phi'' x . \frac{h^2}{1.2} - \phi''' x \cdot \frac{h^3}{1.2.3} + \ldots (4)$$

In these equations $u = \phi x$, and if we transpose this term we shall have

$$u' - u = \phi' x.h + \phi'' x . \frac{h^2}{1.2} + \phi''' x \cdot \frac{h^3}{1.2.3} + \ldots (5)$$

$$u' - u = -\phi' x.h + \phi'' x . \frac{h^2}{1.2} - \phi''' x . \frac{h^3}{1.2.3} + \ldots (6)$$

In these equations $\phi' x$, $\phi'' x$, $\phi''' x$, &c., are finite quantities; and if we take h very small, $\phi' x.h$ will be of the 1st order of small quantities, or infinitesimals, and $\phi'' x \times (h^2 \div 1.2)$, of the 2nd order, &c., so that $\phi'' x \times (h^2 \div 1.2)$ is infinitely small as compared with $\phi' x.h$, and $\phi''' x \times (h^3 \div 1.2.3)$ is infinitely small as compared with $\phi'' x \times (h^2 \div 1.2)$, &c. (See Prof. Ficklin's *Complete Algebra*, p. 129.) Now if u, or its equal the polynomial, has its greatest value, or is a maximum, then u', which is found by increasing or diminishing x by a quantity h, no matter how small, is lees than u; and if u is the least possible, or a minimum, then u' is greater than u. We hence see that the infinitesimal value of h, which we have supposed, is all that we need consider in this demonstration. We can easily see, then, that when h is thus small, the term $\phi' x.h$ will be *greater* than *all* the other terms; for the number of terms in the second members of (5) and (6), is limited, so that their number cannot compensate for their smallness. In the case of a maximum $u' - u$ is negative; and in the case of a minimum $u' - u$ is positive. But because $\phi' x.h$ is greater than all the other terms in the second members of (5) and (6), the *sign* of this term will determine the *sign* of the second member; and from (5) we see that when h is positive, supposing $\phi' x$ positive, the second member is positive, and hence $u' - u$ is positive for a maximum; and from (6) we see that $u' - u$ is negative if h is negative; that is, the second member, or $u' - u$ changes sign with h. But if u is a maximum, whether x be increased or diminished, $u' - u$ is *negative*. To satisfy all these conditions we must make $\phi' x = 0$. For a minimum we must have in all cases, $u' - u$ a *positive* quantity, and in order to secure this, we must also have $\phi' x = 0$.

The term $\phi'' x \times (h^2 \div 1.2)$, which is greater than all the remaining terms, has the same sign whether h be plus or minus. For a maximum we see further, then, that $\phi'' x$ must be *negative*; and for a minimum, $\phi'' x$ must be *positive*.

Our conditions, then, are, for a maximum,

$\phi' x = 0$, and $\phi'' x$ negative, and

$\phi' x = 0$, and $\phi'' x$ positive,

for a minimum.

Now $\phi' x$ is called the first derived polynomial, or function, and $\phi'' x$, the second derived polynomial, &c. (See Ficklin's *Com. Algebra*, p. 400, where he uses fx instead of ϕx.) To find what value of x will render u a maximum, put the first derived polynomial equal to 0, and find the value of x from the resulting equation. Substitute these values, in succession, in the second derived polynomial, and all the values that give a *negative* result indicate so many maxima; and all that give a *positive* result, indicate so many minima. If any of the values of x render $\phi'' x = 0$, then, in order that such values of x may produce maximum or minimum values of u, we must also have $\phi''' x = 0$, for $\phi''' x \times (h^3 \div 1.2.3)$ changes sign with h; and to determine whether we have a maximum or a minimum, we must substitute the values of x in $\phi'''' x$, which will be negative for a maximum and positive for a minimum. For other cases we proceed in a similar manner with the other terms.

This demonstration will apply to any function of x from which we can find $\phi' x$, $\phi'' x$, &c. If the student masters this demonstration, he will find it of great service to him when he takes up the Calculus, which at once furnishes the means of finding $\phi' x$, $\phi'' x$, &c.

Illustrative problem:—Divide a in to two such parts that the square of the first multiplied by the second shall be a maximum.

Let x be the first part, then $a - x$ is the second, and $u = x^2(a - x) = ax^2 - x^3 = \phi x$ is to be a maximum. In this case $\phi' x = 2ax - 3x^2 = 0$, $\phi'' x = 2a - 6x$.

$2ax - 3x^2 = x(2a - 3x) = 0$; $\therefore x = 0$, and $2a - 3x = 0$, $x = \frac{2}{3}a$. If we put the value $x = 0$, in $\phi'' x$, we have $2a$ a positive quantity; and if we put the value $x = \frac{2}{3}a$ in $\phi'' x$ we have $\phi'' x = 2a - 4a = -2a$, a negative quantity. Therefore, the first is a minimum, and the second a maximum.

2. The student will see that the difficulty consists mainly in finding $\phi' x$ and $\phi'' x$, and if we could readily find these we could solve many problems in maxima and minima without the Calculus. In many cases it is unnecessary to find $\phi'' x$, since we can easily satisfy ourselves whether we have a maximum or a minimum. The following process, which I think the student of algebra can easily understand, will enable us to find both $\phi' x$ and $\phi'' x$, and more terms, if we choose, in algebraic functions, or quantities.

If ϕx be an algebraic function of x, then by writing $x + h$ for x, and developing the resulting quantity according to the ascending or positive powers of h, as far as the first and the second power of h, the coefficient of h will be $\phi' x$ (Eqs. (3) and (4)), and that of h^2 will be $\phi'' x$. This development can be effected by the Binomial Theorem, or by Indeterminate Coefficients.

Suppose we have

$$u = \phi x = x\sqrt{(1 - x^2)} + \frac{\sqrt{(1 - x)}}{x}$$

and we wish to find a value of x that will render u a maximum or a minimum. Put $x = x + h$, then

$$u' = (x + h)\sqrt{(1 - x^2 - 2xh - h^2)} + \sqrt{(1 - x - h)}\,(x + h)^{-1}$$
$$= u + \phi' xh \text{ to the first power of } h. \text{ Then}$$

$$u' = x\left(1 + \frac{h}{x}\right)\sqrt{(1 - x^2)}\sqrt{\left(1 - \frac{2xh + h^2}{1 - x^2}\right)}$$
$$+ \frac{1}{x}\sqrt{(1 - x)}\sqrt{\left(1 - \frac{h}{1 - x}\right)}\left(1 + \frac{h}{x}\right)^{-1}$$

$$= x\sqrt{(1-x^2)}\left(1+\frac{h}{x}-\frac{xh}{1-x^2}\right)+\frac{1}{x}\cdot\sqrt{(1-x^2)}\left(1-\frac{h}{2(1-x)}-\frac{h}{x}\right)$$

$$= u + h\left(\sqrt{(1-x^2)}-\frac{x^2}{\sqrt{(1-x^2)}}-\frac{1}{2x\sqrt{(1-x)}}-\frac{\sqrt{(1-x)}}{x^2}\right).$$

We must now put the coefficient of h equal to 0, and we shall have

$$\sqrt{(1-x^2)}-\frac{x}{\sqrt{(1-x^2)}}-\frac{1}{2x\sqrt{(1-x)}}-\frac{\sqrt{(1-x)}}{x^2}=0.$$

The solution of this equation will make known the required values of x Take the equation $u = xe^{-x}$, then

$$u' = (x+h)e^{-x+h} = (x+h)e^{-x}.e^{-h} = e^{-x}(x+h)(1-h+\tfrac{1}{2}h^2)$$

$$= e^{-x}(x-hx+\tfrac{1}{2}h^2x+h-h^2) = u+\psi'x.h+\tfrac{1}{2}\psi''x.h^2.$$

$$\therefore e^{-x}(1-x)=0,\ x=1.$$

Since $\psi''x = e^{-x}(\frac{1}{2}x-1)$, the value of $x = 1$, gives $\psi''x = e^{-1}(\frac{1}{2}-1) = -\frac{1}{2}e^{-1}$, a negative quantity, so that this value of x makes $u = e^{-1}$, a maximum.

We could easily extend this demonstration to the case of two independent variables.

ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

BY G. W. HILL.

In nearly all treatises on algebra the solution of these equations is presented as accomplished by the aid of analytical artifices, which one seems, by some happy hazard, to have stumbled upon. No doubt the processes were found in this manner by the original discoverers, Tartaglia, Carden and Ferrari. But, for many reasons, it would be better to treat the subject as one demanding invention rather than artifice. The equations can, as it were, be interrogated and compelled to yield up their secrets, if they have any.

To say that an equation is solvable algebraically, is to say that an algebraic expression can be found equivalent to the general root, that is, one involving a finite number of the operations of addition, subtraction, multiplication, division and the extraction of roots of prime degree. If the expression does not involve the last mentioned operation, it is called rational, and if free from the two last, integral.